CLT proof

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WLLN $X_1, \dots X_n$ iid $EX_i = \mu$, $\mathrm{var}X_i = \sigma^2 < \infty$

define $\bar{X} = \frac{1}{n}\sum X_i$ (average of a R.V.)

then $\lim_{n\to\infty} P(|\bar{X}_n - \mu| > \epsilon) = 0$

$\bar{X}_n \xrightarrow{P} \mu$ ☆

proof. cheby chev's

$$P(|\bar{X}_n - \mu| > \epsilon) = P(|\bar{X}_n - \mu|^2 > \epsilon^2) \le \frac{E(\bar{X}_n - \mu)^2}{\epsilon^2}$$

$$= \frac{\mathrm{var}(\bar{X}_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \to 0 \quad n\to\infty$$

consistency of S^2 $\quad P(|S_n^2 - \sigma^2| > \epsilon) \le \frac{E(|S_n^2 - \sigma^2|)}{\epsilon^2} = \frac{\mathrm{var}(S_n^2)}{\epsilon^2}$

$S_n^2 \xrightarrow{P} \sigma^2$ if $\mathrm{var}(S_n^2) \to 0$

CLT Thm. sample mean ~ Normal dist

$X_1, X_2, \dots X_n$ iid R.V.s. finite μ & σ^2, MGF exist. ☆

① $\sum X_i \sim N(n\mu, n\sigma^2) \Rightarrow \frac{\sum X_i - n\mu}{\sqrt{n}\sigma} \sim N(0,1)$

② $\frac{1}{n}\sum X_i = \bar{X}_n \sim N(\mu, \frac{\sigma^2}{n}) \Rightarrow \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \sim N(0,1)$

① & ② equivalent

in practice, we only observe one realization → one $\bar{X}_n$.

Proof. show $\underline{\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}}$ MGF $= e^{\frac{t^2}{2}}$ N(0,1) MGF.

没 standard normal easier to work with.

$$M_{\frac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma}}(t) = M_{\sum_{i=1}^n Y_i/\sqrt{n}}(t) \qquad Y_i = \frac{X_i - \mu}{\sigma} \Rightarrow \sum Y_i = \frac{\sum X_i - n\mu}{\sigma}$$

$$\frac{\sqrt{n}(\frac{1}{n}\sum X_i - \mu)}{\sigma} \Downarrow \frac{\frac{1}{\sqrt{n}}\sum X_i - \sqrt{n}\mu}{\sigma} = \frac{\sum X_i - n\mu}{\sqrt{n}\,\sigma}$$

$= [M_{Y_i}(\frac{t}{\sqrt{n}})]^n$ sub in

$$M_{\frac{\sum Y_i}{\sqrt{n}}}(t) = \left[M_{\frac{Y_i}{\sqrt{n}}}(t)\right]^n = \left[M_{Y_i}(\frac{t}{\sqrt{n}})\right]^n$$

sub in

$$M_{\frac{Y_i}{\sqrt{n}}}(t) = E(e^{t/\sqrt{n}\cdot Y_i}) = M_{Y_i}(\frac{t}{\sqrt{n}})$$

→ 将 $M_{Y_i}(\frac{t}{\sqrt{n}})$ expand around 0. → 我觉得是因为在0处展开导数和 Moment 有联系, $M_Y^{(1)}(0) = EX$, $M_Y^{(2)}(0) = EX^2$ 所以在0处展开.

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + (x-x_0)^2\frac{f''(x_0)}{2!} + (x-x_0)^3\frac{f'''(x_0)}{3!} + \dots$$

$$= \sum_{k=0}^{\infty} f^{(k)}(x_0)\frac{(x-x_0)^{(k)}}{k!}$$

→ $f^{(0)}(x_0) = f(x_0)$ 0次导为原函数

$$M_{Y_i}(\frac{t}{\sqrt{n}}) = M_{Y_i}(0) + \frac{t}{\sqrt{n}}\cdot \underbrace{M_Y^{(1)}(0)}_{EX=0} + \left(\frac{t}{\sqrt{n}}\right)^2 \frac{M_Y^{(2)}(0)}{2!}$$ ☆

$\to EX^2 = 1$

$M_{Y_i}(t) = e^{\frac{t^2}{2}}$

MGF of standard normal with $t=0$. ☆

$$= e^{\frac{t^2}{2}}\Big|_{t=0} + (\frac{t}{\sqrt{n}})^2/2!$$

$$= 1 + (\frac{t}{\sqrt{n}})^2/2 + \text{Remainders.}$$

→ $\sum_{k=3}^{\infty} M_Y^{(k)}(0)\cdot\frac{(t/\sqrt{n})^k}{k!} = 0$ 待补充

$$= 1 + \frac{t^2}{2n} + \text{remainders}$$

$$[M_Y(\frac{t}{\sqrt{n}})]^n = (1 + \frac{t^2}{2n})^n$$

$$= (1 + \frac{t^2/2}{n})^n$$

$$\therefore \lim_{n\to\infty} M_Y(\frac{t}{\sqrt{n}})^n = \lim_{n\to\infty}(1 + \frac{t^2/2}{n})^n = e^{\frac{t^2}{2}}$$

Slutsky's theorem.

$X_n \xrightarrow{D} X$

$Y_n \xrightarrow{P} a$ (constant)

$\to X_nY_n \xrightarrow{D} aX$

$\to X_n + Y_n \xrightarrow{D} X + a.$

eg. by CLT $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \xrightarrow{D} N(0,1)$

continuous mapping.

$\boxed{\text{if } \lim_{n\to\infty} \mathrm{var}\, S_n^2 \to 0} \to \boxed{S_n^2 \to \sigma^2} \to \boxed{\frac{\sigma}{S_n} \to 1}$

$\to P(|S_n^2 - \sigma^2| > \epsilon) \le \frac{\mathrm{var}(S_n^2) \to 0}{\epsilon^2} = 0.$

$\therefore$ ① $\frac{\sigma}{S_n} \xrightarrow{P} 1$ (by defn)

② $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{D} N(0,1)$

Slutsky's ① × ② $\frac{\sqrt{n}(\bar{X}_n-\mu)}{S_n} \xrightarrow{D} N(0,1)$

这让我想到了t分布 $\bar{X} \sim N$, $S_n \sim \chi^2$

$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \sim N(0,1)$

$S^2/\sigma^2 \sim \chi^2(n-1)$

Scaled ?